

A POLYNOMIAL ISOPERIMETRIC INEQUALITY FOR $SL(n, \mathbb{Z})$

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ABSTRACT. We prove that when $n \geq 5$, the Dehn function of $SL(n, \mathbb{Z})$ is at most quartic. The proof involves decomposing a disc in $SL(n, \mathbb{R})/SO(n)$ into a quadratic number of loops in generalized Siegel sets. By mapping these loops into $SL(n, \mathbb{Z})$ and replacing large elementary matrices by “shortcuts,” we obtain words of a particular form, and we use combinatorial techniques to fill these loops.

1. INTRODUCTION

The Dehn function is a geometric invariant of a space (typically, a riemannian manifold or a simplicial complex) which measures the difficulty of filling closed curves with discs. This can be made into a group invariant by defining the Dehn function of a group to be the Dehn function of a space on which the group acts cocompactly. The choice of space affects the Dehn function, but its rate of growth depends solely on the group.

The study of Dehn functions of lattices in semisimple Lie groups is a natural direction. For cocompact lattices, this is straightforward; such a lattice acts on a non-positively curved symmetric space X , and this non-positive curvature gives rise to a linear or quadratic Dehn function. Non-cocompact lattices have more complicated behavior. The key difference is that if the lattice is not cocompact, it acts cocompactly on a subset of X rather than the whole thing, and the boundary of this subset may contribute to the Dehn function.

In the case that Γ has \mathbb{Q} -rank 1, the Dehn function is almost completely understood, and depends primarily on the \mathbb{R} -rank of G . In this case, Γ acts cocompactly on a space consisting of X with infinitely many disjoint horoballs removed. When G has \mathbb{R} -rank 1, the boundaries of these horoballs correspond to nilpotent groups, and the lattice is hyperbolic relative to these nilpotent groups. The Dehn function of the lattice is thus equal to that of the nilpotent groups, and Gromov showed that unless X is the complex, quaternionic, or Cayley hyperbolic plane, the Dehn function is at most quadratic [9]. If X is the complex or quaternionic hyperbolic plane, the Dehn function is cubic [9, 17]; if X is the Cayley hyperbolic plane, the precise growth rate is unknown, but is at most cubic.

When G has \mathbb{R} -rank 2 and Γ has \mathbb{Q} -rank 1 or 2, Leuzinger and Pittet [14] proved that the Dehn function grows exponentially. As in the \mathbb{R} -rank 1 case, the proof relies on understanding the subgroups corresponding to the removed horoballs, but in this case the subgroups are solvable and have exponential Dehn function. Finally, when G has \mathbb{R} -rank 3 or greater and Γ has \mathbb{Q} -rank 1, Drutu [6] has shown that the boundary of a horoball satisfies a quadratic filling inequality and that Γ enjoys an

“asymptotically quadratic” Dehn function, i.e., its Dehn function is bounded by $n^{2+\epsilon}$ for any $\epsilon > 0$.

When Γ has \mathbb{Q} -rank larger than 1, the geometry of the space becomes more complicated. The main difference is that the removed horoballs are no longer disjoint, so many of the previous arguments fail. In many cases, the best known result is due to Gromov, who sketched a proof that the Dehn function of Γ is bounded above by an exponential function [9, 5.A₇]. A full proof of this fact was given by Leuzinger [12].

In this paper, we consider $\mathrm{SL}(n, \mathbb{Z})$. This is a lattice with \mathbb{Q} -rank $n - 1$ in a group with \mathbb{R} -rank $n - 1$, so when n is small, the methods above apply. When $n = 2$, the group $\mathrm{SL}(2, \mathbb{Z})$ is virtually free, and thus hyperbolic. As a consequence, its Dehn function is linear. When $n = 3$, the result of Leuzinger and Pittet mentioned above implies that the Dehn function of $\mathrm{SL}(3, \mathbb{Z})$ grows exponentially; this was first proved by Epstein and Thurston [7].

Much less is known about the Dehn function for lattices in $\mathrm{SL}(n, \mathbb{Z})$ when $n \geq 4$. By the results of Gromov and Leuzinger above, the Dehn function of any such lattice is bounded by an exponential function, but the Dehn function may be polynomial in many cases. Thurston [7] conjectured that

Conjecture 1. *When $n \geq 4$, $\mathrm{SL}(n, \mathbb{Z})$ satisfies the isoperimetric inequality*

$$\delta_{\mathrm{SL}(n, \mathbb{Z})}(\ell) \lesssim \ell^2.$$

In this paper, we will prove that

Theorem 1. *When $n \geq 5$, $\mathrm{SL}(n, \mathbb{Z})$ satisfies the isoperimetric inequality*

$$\delta_{\mathrm{SL}(n, \mathbb{Z})}(\ell) \lesssim \ell^4.$$

In Section 2, we present some preliminaries, and in Section 3, we sketch an overview of the proof. In Sections 4–7, we prove Theorem 1.

Some of the ideas in this work were inspired by discussions at the American Institute of Mathematics workshop, “The Isoperimetric Inequality for $\mathrm{SL}(n, \mathbb{Z})$,” and the author would like to thank the organizers, Nathan Broaddus, Tim Riley, and Kevin Wortman; and participants, especially Mladen Bestvina, Alex Eskin, Martin Kassabov, and Christophe Pittet. The author would also like to thank Tim Riley and Yves de Cornulier for many helpful conversations while the author was visiting Bristol University and Université de Rennes.

2. PRELIMINARIES

In this section, we recall several facts about $\mathrm{SL}(p, \mathbb{Z})$, $\mathrm{SL}(p, \mathbb{R})$, and about Dehn functions.

We provide only a minimal introduction to Dehn functions here; for a survey with examples, see for instance [2]. The Dehn function is a group invariant which gives one way to describe the difficulty of determining whether a word in a group represents the identity. It can be described both combinatorially and geometrically, and the interaction between these two viewpoints is often crucial. We first give some terminology. If X is a set, and $x_i \in X$ for $1 \leq i \leq n$, we call the formal product $x_1 \dots x_n$ a word in X . Let X^* to be the set of words in $X \cup X^{-1}$, where X^{-1} is the set of formal inverses of elements of X . We denote the empty word by ε . If $w \in X^*$, we can write $w = x_1 x_2 \dots x_n$, and we define the length $\ell(w)$ of w to be n . Note especially that these words are not reduced; that is, x may appear next to

x^{-1} . If $X \subset H$ for some group H , there is a natural evaluation map $X^* \rightarrow H$, and we say that words represent elements of H .

Using these concepts, we can describe the combinatorial Dehn function. If

$$H = \langle h_1, \dots, h_d \mid r_1, \dots, r_s \rangle$$

is a finitely presented group, we can let $\Sigma = \{h_1, \dots, h_d\}$ and consider words in Σ^* . If a word w represents the identity, then there is a way to prove this using the relations. That is, there is a sequence of steps which reduces w to the empty word, where each step is a free expansion (insertion of a subword $x_i^{\pm 1} x_i^{\mp 1}$), free reduction (deletion of a subword $x_i^{\pm 1} x_i^{\mp 1}$), or the application of a relator (insertion or deletion of one of the r_i). We call the number of applications of relators in a sequence its *cost*, and we call the minimum cost of a sequence which starts at w and ending at ε the *filling area* of w , denoted by $\delta_H(w)$. We then define the *Dehn function* of H to be

$$\delta_H(n) = \max_{\ell(w) \leq n} \delta_H(w),$$

where the maximum is taken over words representing the identity. This depends *a priori* on the chosen presentation of H ; we will see that the growth rate of δ_H is independent of this choice. For convenience, if v, w are two words representing the same element of H , we define $\delta_H(v, w) = \delta_H(vw^{-1})$; this denotes the minimum cost to transform v to w .

This can also be interpreted geometrically. If K_H is the *presentation complex* of H (a simply-connected 2-complex whose 1-skeleton is the Cayley graph of H and whose 2-cells correspond to translates of the relators), then w corresponds to a closed curve in the 1-skeleton of K_H . Similarly, the sequence of steps reducing w to the identity corresponds to a homotopy contracting this closed curve to a point. More generally, if X is a riemannian manifold or simplicial complex, we can define the filling area $\delta_X(\gamma)$ of a Lipschitz curve $\gamma : S^1 \rightarrow X$ to be the infimal area of a Lipschitz map $D^2 \rightarrow X$ which extends γ . Then we can define the Dehn function of X to be

$$\delta_X(n) = \sup_{\ell(\gamma) \leq n} \delta_X(\gamma),$$

where the supremum is taken over null-homotopic closed curves. As in the combinatorial case, if β and γ are two curves connecting the same points which are homotopic with their endpoints fixed, we define $\delta_X(\beta, \gamma)$ to be the infimal area of a homotopy between β and γ which fixes their endpoints.

Gromov stated a theorem connecting these two definitions, proofs of which can be found in [2] and [3]:

Theorem 2 (Gromov's Filling Theorem). *If X is a simply connected riemannian manifold or simplicial complex and H is a finitely presented group acting properly discontinuously, cocompactly, and by isometries on M , then $\delta_H \sim \delta_M$.*

Here, \sim is an equivalence relation which requires that δ_H and δ_M have the same growth rate according to the following definition: if $f, g : \mathbb{N} \rightarrow \mathbb{N}$, let $f \lesssim g$ if and only if there is a c such that

$$f(n) \leq cg(cn + c) + c \text{ for all } n$$

and $f \sim g$ if and only if $f \lesssim g$ and $g \lesssim f$. One consequence of Theorem 2 is that the Dehn functions corresponding to different presentations are equivalent under this relation.

We state the following lemma, which is used in the proof of Theorem 2. The lemma follows from the Federer-Fleming Deformation Lemma [8] or from the Cellulation Lemma [2, 5.2.3]:

Lemma 1. *Let H and X be as in the Filling Theorem, and let $f : K_H \rightarrow X$ be an H -equivariant map of a presentation complex for H to X . There is a c such that:*

- (1) *Let $s : [0, 1] \rightarrow X$ connect $f(e)$ and $f(h)$, where e is the identity in H and $h \in H$. There is a word w which represents h and which has length $\ell(w) \leq c\ell(s) + c$. If X is simply connected, then w approximates s in the sense that if $\gamma_w : [0, 1] \rightarrow K_H$ is the curve corresponding to w , then*

$$\delta_X(s, \gamma_w) \leq c\ell(s) + c.$$

- (2) *If w is a word representing the identity in H and $\gamma : S_1 \rightarrow K_H$ is the corresponding closed curve in K_H , then*

$$\delta_H(w) \leq c(\ell(w) + \delta_X(f \circ \gamma)).$$

We now set out notation for $\mathrm{SL}(p)$ and several of its subgroups. In the following, \mathbb{K} represents either \mathbb{Z} or \mathbb{R} ; when it is omitted, we take it to be \mathbb{R} . Let $G = \mathrm{SL}(p, \mathbb{R})$ and let $\Gamma = \mathrm{SL}(p, \mathbb{Z})$. Let z_1, \dots, z_p generate \mathbb{Z}^p , and if $S \subset \{1, \dots, p\}$, let $\mathbb{K}^S = \langle z_s \rangle_{s \in S}$ be a subspace of \mathbb{K}^p . If $q \leq p$, there are many ways to include $\mathrm{SL}(q)$ in $\mathrm{SL}(p)$. Let $\mathrm{SL}(S)$ be the copy of $\mathrm{SL}(\#S)$ in $\mathrm{SL}(p)$ which acts on \mathbb{R}^S and fixes z_t for $t \notin S$. If S_1, \dots, S_n are disjoint subsets of $\{1, \dots, p\}$ such that $\bigcup S_i = \{1, \dots, p\}$, let

$$U(S_1, \dots, S_n; \mathbb{K}) \subset \mathrm{SL}(p, \mathbb{K})$$

be the subgroup of matrices preserving the flag

$$\mathbb{R}^{S_i} \subset \mathbb{R}^{S_i \cup S_{i-1}} \subset \dots \subset \mathbb{R}^p$$

when acting on the right. If the S_i are sets of consecutive integers in increasing order, $U(S_1, \dots, S_n; \mathbb{K})$ is block upper triangular. For example, $U(\{1\}, \{2, 3, 4\}; \mathbb{K})$ is the subgroup of $\mathrm{SL}(4, \mathbb{K})$ consisting of matrices of the form:

$$\begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{pmatrix}.$$

If $d_1, \dots, d_n > 0$, let $U(d_1, \dots, d_n; \mathbb{K})$ be the group of upper block triangular matrices with blocks of the given lengths, so that the subgroup illustrated above is $U(1, 3; \mathbb{K})$. Each group $U(d_1, \dots, d_n; \mathbb{Z})$ is a parabolic subgroup of Γ , and any parabolic subgroup of Γ is conjugate to a unique such group. Let \mathcal{P} be the set of these groups.

We will note some facts about the combinatorial group theory of Γ and its subgroups. Let I be the identity matrix. If $1 \leq i \neq j \leq p$, let $e_{ij}(x) \in \mathrm{SL}(p, \mathbb{Z})$ be the elementary matrix which consists of the identity matrix with the (i, j) -entry replaced by x . Let $e_{ij} := e_{ij}(1)$. There is a finite presentation which has the matrices e_{ij} as generators [16]:

$$(1) \quad \begin{aligned} \mathrm{SL}(p, \mathbb{Z}) &= \langle e_{ij} \mid [e_{ij}, e_{kl}] = I && \text{if } i \neq l \text{ and } j \neq k \\ &[e_{ij}, e_{jk}] = e_{ik} && \text{if } i \neq k \\ &(e_{ij}e_{ji}^{-1}e_{ij})^4 = I \rangle, \end{aligned}$$

where we adopt the convention that $[x, y] = xyx^{-1}y^{-1}$.

We will use a slightly expanded set of generators. Let

$$\Sigma = \Sigma(p) = \{e_{ij} \mid 1 \leq i \neq j \leq p\} \cup D,$$

where D is the set of diagonal matrices in $\mathrm{SL}(p, \mathbb{Z})$. Then there is a finite presentation of $\mathrm{SL}(p, \mathbb{Z})$ with generating set Σ and relations consisting of those in (1) and relations expressing each element of D as a product of elementary matrices. The advantage of this generating set is that if $H = \mathrm{SL}(S, \mathbb{Z})$ or $H = U(S_1, \dots, S_n; \mathbb{Z})$, then H is generated by $\Sigma \cap H$.

The group Γ is a lattice in $G = \mathrm{SL}(p, \mathbb{R})$, and the geometry of G and of the quotient will be important in our proof. We think of G and Γ as acting on the symmetric space on the left. Let $\mathcal{E} = \mathrm{SL}(p, \mathbb{R})/\mathrm{SO}(p, \mathbb{R})$. The tangent space of \mathcal{E} at the identity, $T_I\mathcal{E}$ is isomorphic to the space of symmetric matrices with trace 0. If u^{tr} represents the transpose of u , then we can define an inner product $\langle u, v \rangle = \mathrm{trace}(u^{tr}v)$ on $T_I\mathcal{E}$. Since this is $\mathrm{SO}(p)$ -invariant, it gives rise to a G -invariant riemannian metric on \mathcal{E} . Under this metric, \mathcal{E} is a non-positively curved symmetric space. The lattice Γ acts on \mathcal{E} with finite covolume, but the action is not cocompact. Let $\mathcal{M} := \Gamma \backslash \mathcal{E}$. If $x \in G$, we write the equivalence class of x in \mathcal{E} as $[x]_{\mathcal{E}}$; similarly, if $x \in G$ or $x \in \mathcal{E}$, we write the equivalence class of x in \mathcal{M} as $[x]_{\mathcal{M}}$.

If $g \in G$ is a matrix with coefficients $\{g_{ij}\}$, we define

$$\|g\|_2 = \sqrt{\sum_{i,j} g_{ij}^2},$$

$$\|g\|_{\infty} = \max_{i,j} |g_{ij}|.$$

Note that for all $g, h \in G$, we have

$$\|gh\|_2 \leq \|g\|_2 \|h\|_2$$

$$\|g^{-1}\|_2 \geq \|g\|_2^{1/p}$$

and that there is a c such that

$$c^{-1}d_G(I, g) - c \leq \log \|g\|_2 \leq cd_G(I, g) + c.$$

It will be useful to have a geometric picture of elements of \mathcal{E} and \mathcal{M} . The rows of a matrix in $\mathrm{SL}(p, \mathbb{R})$ give a unit volume basis of \mathbb{R}^p , and we can think of G as the set of such bases. From this viewpoint, $\mathrm{SO}(p)$ acts on a basis by rotating the basis vectors, so \mathcal{E} consists of the set of bases up to rotation. An element of Γ acts by replacing the basis elements by integer combinations of basis elements. This preserves the lattice that they generate, so we can think of $\Gamma \backslash G$ as the set of unit-covolume lattices in \mathbb{R}^p . The quotient \mathcal{M} is then the set of unit-covolume lattices up to rotation. Nearby points in \mathcal{M} or \mathcal{E} correspond to bases or lattices which can be taken into each other by small linear deformations of \mathbb{R}^p .

Finally, we define a subset of \mathcal{E} on which Γ acts cocompactly. Let $\mathcal{E}(\epsilon)$ be the set of points which correspond to lattices with injectivity radius at least ϵ . When $\epsilon \leq 1/2$, this set is contractible and Γ acts on it cocompactly [7]; we call it the *thick part* of \mathcal{E} , and its preimage $G(\epsilon)$ in G the thick part of G . Let $\iota : K_{\Gamma} \rightarrow \mathcal{E}$ be a Γ -equivariant map; if ϵ is sufficiently small, then the image of ι is contained in $\mathcal{E}(\epsilon)$.

3. OVERVIEW OF PROOF

To understand our methods for proving a polynomial Dehn function for Γ , it is helpful to consider a related method for proving an exponential Dehn function. Let $w \in \Sigma^*$ be a word which represents the identity in Γ , so that w corresponds to a closed curve in K_Γ . By abuse of notation, we also call this curve w . We can construct a curve $\alpha : S^1 \rightarrow \mathcal{E}$ which corresponds to w by letting $\alpha = [\iota(w)]_{\mathcal{E}}$. Let $\ell = \ell(\alpha)$ and assume that α is parameterized by length.

Since \mathcal{E} is non-positively curved, we can use geodesics to fill α . If $x, y \in \mathcal{E}$, let $\lambda_{x,y} : [0, 1] \rightarrow \mathcal{E}$ be a geodesic parameterized so that $\lambda_{x,y}(0) = x$, $\lambda_{x,y}(1) = y$, and $\lambda_{x,y}$ has constant speed. We can define a homotopy $h : [0, \ell] \times [0, 1] \rightarrow \mathcal{E}$ by

$$h(x, t) = \lambda_{\alpha(x), \alpha(0)}(t/\ell).$$

Let $D^2 \subset \mathbb{R}^2$ be the disc of radius ℓ centered at the origin and let

$$f(r, \theta) = h(\ell \frac{\theta}{2\pi}, r/\ell)$$

where r and θ are polar coordinates. Since \mathcal{E} is non-positively curved, this map is Lipschitz and its Lipschitz constant $\text{Lip}(f)$ is bounded independently of α ; in particular, it has area $O(\ell^2)$. Furthermore, the image of f is contained in a ball around $[I]_{\mathcal{E}}$ of radius ℓ .

Since Γ does not act cocompactly on \mathcal{E} , this filling does not directly correspond to an efficient filling of w in K_Γ . To construct a filling in K_Γ , we will need a map $\rho : \mathcal{E} \rightarrow \Gamma$. We can construct one from a fundamental set for the action of Γ on \mathcal{E} ; we let \mathcal{S} be a Siegel set (see Sec. 4) and define ρ so that for all $x \in \mathcal{E}$, $x \in \rho(x)\mathcal{S}$. Since \mathcal{M} is not compact, this map is not a quasi-isometry; if $x \in \mathcal{E}$ is deep in the cusp of \mathcal{M} , then small changes in x can result in large changes in $\rho(x)$. On the other hand, the injectivity radius in the cusp shrinks at most exponentially with the distance from a basepoint. That is, there is a c such that if $x \in B_r(I) \subset \mathcal{E}$, and $d_{\mathcal{E}}(x, y) < \exp(-cr)$, then $d_\Gamma(\rho(x), \rho(y)) \leq c$.

Our basic technique is to construct a triangulation τ of the disc, and use f as a template for a map $\bar{f} : \tau \rightarrow K_\Gamma$. We will construct $\bar{f} : \tau \rightarrow K_\Gamma$ one dimension at a time. Let τ be a triangulation of D^2 with $O(e^{2c\ell})$ cells such that the image of each cell under f has diameter at most $e^{-c\ell}$. If x and y are vertices of an edge of τ , then

$$d_\Gamma(\rho(f(x)), \rho(f(y))) \leq c,$$

where d_Γ is the word metric on Γ given by the generating set Σ . Let $\bar{f}_0 : \tau^{(0)} \rightarrow K_\Gamma$ be given by $\bar{f}_0(x) = \rho(f(x))$ for all x (where we identify elements of Γ with the corresponding vertices in K_Γ).

To construct $\bar{f}_1 : \tau^{(1)} \rightarrow K_\Gamma$, we must find words in Σ^* which connect the images of adjacent vertices of τ ; that is, for each edge $e = (x, y)$, we must find a word in Σ^* representing $\bar{f}_0(x)^{-1}\bar{f}_0(y)$. Since $\bar{f}_0(x)^{-1}\bar{f}_0(y)$ is a bounded element of Γ , we choose $\bar{f}_1(e)$ to be a word of length at most c .

Finally, we construct \bar{f} on the triangles of τ . If Δ is a triangle of τ , then $\bar{f}_1(\partial\Delta)$ corresponds to a word of length at most $3c$ which represents the identity. Since K_Γ is simply connected, each such word can be filled by a disc of area at most $\delta_\Gamma(3c)$. This results in a map \bar{f} of area $O(e^{2c\ell})$. The boundary of \bar{f} is not quite w , but it remains a bounded distance from w , and there is a homotopy between the two of area $O(\ell)$. Thus $\delta_\Gamma(w) = O(e^{2c\ell})$, and

$$\delta_\Gamma(\ell) \lesssim e^\ell,$$

as desired.

We will prove a polynomial bound with a similar scheme. The main difference is that we construct τ by dividing D^2 into $O(\ell^2)$ triangles of diameter ≤ 1 instead of exponentially many triangles of exponentially small diameter. We define ρ and \bar{f}_0 as described above, but it is no longer the case that if x and y are connected by an edge, then $d_\Gamma(\bar{f}_0(x), \bar{f}_0(y)) < c$. In Section 5, we use the geometry of \mathcal{M} to show instead that $\bar{f}_0(x)^{-1}\bar{f}_0(y)$ is the product of a block-diagonal element of Γ with bounded coefficients and a unipotent element with at most exponentially large coefficients.

Because $\bar{f}_0(x)^{-1}\bar{f}_0(y)$ is no longer a bounded element of Γ , we must change the way we define \bar{f}_1 as well. In Section 6, we will define a normal form for block upper-triangular matrices with bounded block-diagonal part and exponentially large unipotent part. We will replace edges of τ with words in this normal form which have length $O(\ell)$.

Finally, we construct \bar{f} by extending \bar{f}_1 to the 2-cells of τ . The boundary of each 2-cell is a product of three words in normal form and has length $O(\ell)$; in Section 7, we will show that such words can be filled with discs of area $O(\ell^2)$. Since there are $O(\ell^2)$ such triangles to fill, this method will give an ℓ^4 upper bound on the Dehn function.

4. CONSTRUCTING A FUNDAMENTAL SET

In this section, we will define \mathcal{S} , a fundamental set for Γ . Let $\text{diag}(t_1, \dots, t_p)$ be the diagonal matrix with entries (t_1, \dots, t_p) . Let A be the set of diagonal matrices in G and let

$$A_\epsilon^+ = \{\text{diag}(t_1, \dots, t_p) \mid \prod t_i = 1, t_i > 0, t_i \geq \epsilon t_{i+1}\}.$$

Let N be the set of upper triangular matrices with 1's on the diagonal and let N^+ be the subset of N with off-diagonal entries in the interval $[-1/2, 1/2]$. Translates of the set $N^+A_\epsilon^+$ are known as Siegel sets. The following properties of Siegel sets are well known (see for instance [1]).

Lemma 2.

There is an $1 > \epsilon_S > 0$ such that if we let

$$\mathcal{S} := [N^+A_{\epsilon_S}^+]_{\mathcal{E}} \subset \mathcal{E},$$

then

- $\Gamma\mathcal{S} = \mathcal{E}$.
- *There are only finitely many elements $\gamma \in \Gamma$ such that $\gamma\mathcal{S} \cap \mathcal{S} \neq \emptyset$.*

We define $A^+ := A_{\epsilon_S}^+$. Translates of \mathcal{S} cover all of \mathcal{E} , so we can define a map $\rho : \mathcal{E} \rightarrow \Gamma$ such that $\rho(\mathcal{S}) = I$ and $x \in \rho(x)\mathcal{S}$ for all x . As in Section 3, we define $\bar{f}_0 : \tau^{(0)} \rightarrow K_\Gamma$ by $\bar{f}_0(x) = \rho(f(x))$.

The inclusion $A^+ \hookrightarrow \mathcal{S}$ is a Hausdorff equivalence:

Lemma 3. *Give A the riemannian metric inherited from its inclusion in G , so that*

$$d_A(\text{diag}(d_1, \dots, d_p), \text{diag}(d'_1, \dots, d'_p)) = \sqrt{\sum_{i=1}^p \left| \log \frac{d'_i}{d_i} \right|^2}.$$

- *There is a c such that if $x \in \mathcal{S}$, then $d_{\mathcal{E}}(x, [A^+]_{\mathcal{E}}) \leq c$.*

- If $x, y \in A^+$, then $d_A(x, y) = d_S(x, y)$.

Proof. For the first claim, note that if $x = [na]_{\mathcal{E}}$, then $x = [a(a^{-1}na)]_{\mathcal{E}}$, and $a^{-1}na \in N$. Furthermore,

$$\|a^{-1}na\|_{\infty} \leq \epsilon_S^p,$$

so

$$d_{\mathcal{E}}([x]_{\mathcal{E}}, [a]_{\mathcal{E}}) \leq d_G(I, a^{-1}na)$$

is bounded independently of x .

For the second claim, we clearly have $d_A(x, y) \geq d_S(x, y)$. For the reverse inequality, it suffices to note that the map $\mathcal{S} \rightarrow A^+$ given by $na \mapsto a$ for all $n \in N^+$, $a \in A^+$ is distance-decreasing. \square

Siegel conjectured that the quotient map from \mathcal{S} to \mathcal{M} is also a Hausdorff equivalence, that is:

Theorem 3. *There is a c such that if $x, y \in \mathcal{S}$, then*

$$d_S(x, y) - c \leq d_{\mathcal{M}}([x]_{\mathcal{M}}, [y]_{\mathcal{M}}) \leq d_S(x, y)$$

Proofs of this conjecture can be found in [13, 10, 5]. One consequence is that A^+ is Hausdorff equivalent to \mathcal{M} , and it will be helpful to have a map $\phi_{\mathcal{M}} : \mathcal{M} \rightarrow A^+$ which realizes this Hausdorff equivalence. Ji and MacPherson [11] used precise reduction theory to define such a map in a more general setting. In the special case that $G = \mathrm{SL}(n)$ and $\Gamma = \mathrm{SL}(n, \mathbb{R})$, their map and the map $\phi_{\mathcal{M}}$ that we will define differ by a bounded distance.

Any point $x \in \mathcal{E}$ can be written as $x = [\gamma na]_{\mathcal{E}}$ for some $\gamma \in \Gamma$, $n \in N^+$ and $a \in A^+$ in at most finitely many ways. These decompositions have the following property:

Corollary 4 (see [11], Lemmas 5.13, 5.14). *There is a constant c_{ϕ} such that if $x, y \in \mathcal{E}$, $\gamma, \gamma' \in \Gamma$, $n, n' \in N^+$ and $a, a' \in A^+$ are such that $x = [\gamma na]_{\mathcal{E}}$ and $y = [\gamma' n' a']_{\mathcal{E}}$, then*

$$|d_{\mathcal{M}}([x]_{\mathcal{M}}, [y]_{\mathcal{M}}) - d_A(a, a')| \leq c_{\phi}.$$

In particular, if $[\gamma na]_{\mathcal{E}} = [\gamma' n' a']_{\mathcal{E}}$, then

$$d_A(a, a') \leq c_{\phi}.$$

Proof. Without loss of generality, we may assume that $\gamma = \gamma' = I$. Let c be as in Theorem 3 and let c' be as in Lemma 3, so that

$$d_{\mathcal{M}}([na]_{\mathcal{M}}, [a]_{\mathcal{M}}) \leq d_{\mathcal{E}}([na]_{\mathcal{E}}, [a]_{\mathcal{E}}) \leq c'.$$

Then

$$\begin{aligned} |d_{\mathcal{M}}([x]_{\mathcal{M}}, [y]_{\mathcal{M}}) - d_A(a, a')| &= |d_{\mathcal{M}}([na]_{\mathcal{M}}, [n' a']_{\mathcal{M}}) - d_A(a, a')| \\ &\leq c + |d_S([na]_{\mathcal{E}}, [n' a']_{\mathcal{E}}) - d_S([a]_{\mathcal{E}}, [a']_{\mathcal{E}})| \\ &\leq c + 2c'. \end{aligned}$$

\square

We now define $\phi_{\mathcal{M}}$. Any point $x \in \mathcal{E}$ can be uniquely written as $x = [\rho(x)na]_{\mathcal{E}}$ for some $n \in N^+$ and $a \in A^+$. Let $\phi : \mathcal{E} \rightarrow A^+$ be the map $[\rho(x)na]_{\mathcal{E}} \mapsto a$. This is not quite Γ -equivariant, but we can still define a map $\phi_{\mathcal{M}} : \mathcal{M} \rightarrow A^+$ by choosing a lift $\tilde{x} \in \mathcal{E}$ for all $x \in \mathcal{M}$ and defining $\phi_{\mathcal{M}}(x) = \phi(\tilde{x})$. By the corollary, $\phi_{\mathcal{M}}(x)$ is a Hausdorff equivalence with constant c_{ϕ} .

5. BOUNDING GROUP ELEMENTS CORRESPONDING TO EDGES

In this section, we will restrict the possible values of $\rho(x)^{-1}\rho(y)$ when $x, y \in \mathcal{E}$ and $d_{\mathcal{E}}(x, y) \leq 1$. This is a key step in extending \bar{f}_0 to the 1-skeleton of τ .

The possible values of $\rho(x)^{-1}\rho(y)$ depend on $\phi(x)$. We will construct a cover of A^+ by sets corresponding to parabolic subgroups so that the possible values of $\bar{f}_0(x)^{-1}\bar{f}_0(y)$ depend on which set $\phi(x)$ falls into. If $P = U(d_1, \dots, d_r)$, where $\sum d_i = p$, let $s_i = \sum_{j=1}^i d_j$ for $0 \leq i \leq r$. Let

$$X_P(t) = \{\text{diag}(a_1, \dots, a_p) \in A \mid ta_{i+1} < a_i \text{ if and only if } i \in \{s_1, \dots, s_{r-1}\}\}.$$

These sets partition A^+ into 2^{p-1} disjoint subsets.

If $\phi(x) \in X_P(t)$ for some sufficiently large t , then the geometry of the lattice corresponding to x is quite distinctive. Recall that if $\tilde{x} \in G$ is a representative of $x \in \mathcal{E}$, then we can construct a lattice $\mathbb{Z}^p \tilde{x} \subset \mathbb{R}^p$, and different representatives of x correspond to rotations of $\mathbb{Z}^p \tilde{x}$. Let

$$V(x, r) = \{v \in \mathbb{Z}^p \mid \|v\tilde{x}\|_2 \leq r\};$$

this corresponds to the subspace of the lattice generated by vectors of length at most r , and is independent of the choice of \tilde{x} . As such, $V(x, r)$ is Γ -equivariant: if $\gamma \in \Gamma$, then $V(\gamma x, r) = V(x, r)\gamma^{-1}$. In many cases, $\phi(x)$ and $\rho(x)$ determine $V(x, r)$. Let $z_1, \dots, z_p \in \mathbb{Z}^p$ be the standard generating set of \mathbb{Z}^p , and let $Z_j = \langle z_j, \dots, z_p \rangle$.

Lemma 5. *There is a $c_V > 1$ such that if $x \in \mathcal{E}$, $\phi(x) = \text{diag}(a_1, \dots, a_p)$, and*

$$a_{j+1}c_V < r < c_V^{-1}a_j,$$

then $V(x, r) = Z_j\rho(x)^{-1}$.

Proof. It suffices to show that if the hypotheses hold and $x \in \mathcal{S}$, then $V(x, r) = Z_j$. There is an $n = \{n_{ij}\} \in N^+$ such that $x = [n\phi(x)]_{\mathcal{E}}$, and if $\tilde{x} = n\phi(x)$, then

$$\begin{aligned} z_j \tilde{x} &= z_j n \phi(x) \\ &= a_j z_j + \sum_{i=j+1}^p n_{ji} z_i a_i. \end{aligned}$$

Since $|n_{ji}| \leq 1/2$ when $i > j$ and $a_{i+1} \leq a_i \epsilon_S^{-1}$, we have

$$\|z_j \tilde{x}\|_2 \leq a_j \sqrt{p} \epsilon_S^{-p},$$

so

$$V(x, a_j \sqrt{p} \epsilon_S^{-p}) \supset Z_j.$$

On the other hand, if $v \notin Z_j$, then $v = \sum_i v_i z_i$ for some $v_i \in \mathbb{Z}$. Let k be the smallest k such that $v_k \neq 0$; by assumption, $k < j$. The z_k -coordinate of $v\tilde{x}$ is $v_k a_k$, so

$$\|v n \phi(x)\|_2 \geq |a_k| > a_{j-1} \epsilon_S^p$$

and thus if $t < a_{j-1} \epsilon_S^p$, then $V(x, t) \subset Z_j$. Therefore, if

$$a_j \sqrt{p} \epsilon_S^{-p} \leq t < a_{j-1} \epsilon_S^p,$$

then $V(\tilde{x}, t) = Z_j$. □

In particular, if $\phi(x) \in X_P(2c_V^2)$, then $a_{s_i+1}c_V < c_V^{-1}a_{s_i}$ and we can find r_i such that $V(x, r_i) = Z_{s_i}\rho(x)^{-1}$.

Let M_P be the subgroup of P consisting of block diagonal matrices, so that M_P contains $\mathrm{SL}(d_1) \times \cdots \times \mathrm{SL}(d_n)$ as a finite index subgroup. Let $N_P \subset P$ be the subgroup of block upper triangular matrices whose diagonal blocks are the identity matrix. Any element $z \in P$ can be uniquely decomposed as a product $z = nm$, where $n \in N_P$ and $m \in M_P$; we call m the P -reductive part of z and n the P -unipotent part. We will show that if $d(x, y) \leq 1$, then $z = \rho(x)^{-1}\rho(y) \in P$ for some P , where the P -reductive part of z has coefficients bounded independently of ℓ and the P -unipotent part has coefficients at most exponential in ℓ .

Lemma 6. *Let $p \geq 3$. There is a $t_0 > 0$ and a $c_\rho > 0$ such that for all $P \in \mathcal{P}$ and all $x, y \in \mathcal{E}$ such that $\phi(x) \in X_P(t_0)$ and $d_{\mathcal{E}}(x, y) \leq 1$, we can decompose $\rho(x)^{-1}\rho(y)$ as a product $\rho(x)^{-1}\rho(y) = nm$, where $n \in N_P(\mathbb{Z})$, $m \in M_P(\mathbb{Z})$, $d_\Gamma(I, m) < c_\rho$, and $\|n\|_2 \leq c_\rho e^{c_\rho d_{\mathcal{E}}(x, [I]_{\mathcal{E}})}$.*

Proof. Let

$$t_0 = 2 \exp(4(c_\phi + 1))c_V^2,$$

let $P = U(d_1, \dots, d_n)$, and let x and y be as in the hypothesis of the lemma. By translating x and y by $\rho(x)^{-1}$, we may assume that $x \in \mathcal{S}$ and thus $\rho(x) = I$. We first claim that $\rho(y) \in P(\mathbb{Z})$.

Let a_i, a'_i be such that $\phi(x) = \mathrm{diag}(a_1, \dots, a_p)$ and let $\phi(y) = \mathrm{diag}(a'_1, \dots, a'_p)$. Let $r_i = a_{s_i}\sqrt{t_0}$. We claim that for all i ,

$$Z_{s_i} = V(x, r_i) = V(y, r_i) = Z_{s_i}\rho(y)^{-1}.$$

The fact that $Z_{s_i} = V(x, r_i)$ follows from Lemma 5; in fact, $V(x, e^{-1}r_i) = V(x, er_i) = Z_{s_i}$.

Since $d_{\mathcal{E}}(x, y) \leq 1$, the lattices corresponding to x and y only differ by a small deformation; this deformation can change the length of a vector in the lattice by at most a factor of e . Thus, if $v \in \mathbb{Z}^p$, then

$$e^{-1} \leq \frac{\|v\tilde{x}\|_2}{\|v\tilde{y}\|_2} \leq e,$$

and in particular,

$$V(x, e^{-1}r_i) \subset V(y, r_i) \subset V(x, er_i).$$

Since the outer two sets are equal, we have $V(x, r_i) = V(y, r_i)$.

Finally, we need to show that $V(y, r_i) = Z_{s_i}\rho(y)^{-1}$. By the lemma, it suffices to show that $a'_{s_i}c_V < r_i < c_V^{-1}a'_{s_i}$. By Corollary 4, we know that $d_A(\phi(x), \phi(y)) \leq c_\phi + 1$; in particular,

$$\left| \log \frac{a_i}{a'_i} \right| \leq c_\phi + 1,$$

and $a'_{s_i}c_V < r_i < c_V^{-1}a'_{s_i}$ as desired. Thus $Z_{s_i} = Z_{s_i}\rho(y)^{-1}$ for all i , so $\rho(y) \in P$.

We decompose $\rho(y)$ as a product $\rho(y) = n_y m_y$, where $n_y \in N_P(\mathbb{Z})$ and $m_y \in M_P$ consists of the diagonal blocks of $\rho(y)$. We will bound m_y by constructing a map from \mathcal{E} to a product of symmetric spaces.

Let $A_P \subset P$ be the subgroup consisting of diagonal matrices whose diagonal blocks are scalar matrices with positive coefficients; this is isomorphic to $(\mathbb{R}^+)^{n-1}$. The parabolic subgroup P can be uniquely decomposed according to the Langlands decomposition as $P = N_P M_P A_P$, and we can define a map $\mu : P \rightarrow M_P$ so that

if $g = nma$, where $n \in N_P$, $m \in M_P$, and $a \in A_P$, then $\mu(g) = m$. Furthermore, since M_P normalizes A_P and N_P , this is a homomorphism.

This descends to a map on symmetric spaces; if we let $K_P = \mathrm{SO}(p) \cap P$, we get a map $\mu_{\mathcal{E}} : \mathcal{E} \rightarrow M_P/K_P$. This map is Lipschitz. Furthermore, if $p \in P$, $x \in \mathcal{E}$, then $\mu_{\mathcal{E}}(px) = \mu(p)\mu_{\mathcal{E}}(x)$.

This map can be interpreted geometrically. Note that M_P/K_P is a product of symmetric spaces of lower dimensions, so $\mu_{\mathcal{E}}$ breaks a lattice in \mathbb{R}^p into lattices in lower-dimensional subspaces. Let $V_0 = \{0\}$ and $V_i = Z_{s_i}$, so that P preserves the flag $V_0 \subset \cdots \subset V_n = \mathbb{Z}^p$. Then if $g \in G$ (not necessarily parabolic) is a representative of x , then $V_i g / V_{i-1} g$ is a d_i -dimensional lattice in $(\mathbb{R} \otimes V_i)g / (\mathbb{R} \otimes V_{i-1})g$. This lattice generally does not have unit covolume, but we can rescale and possibly reflect it to a unit-covolume lattice. These lattices correspond to a point in

$$M_P/K_P = \mathrm{SL}(d_1, \mathbb{R})/\mathrm{SO}(d_1) \times \cdots \times \mathrm{SL}(d_n, \mathbb{R})/\mathrm{SO}(d_n),$$

and this point is $\mu_{\mathcal{E}}(x)$.

The group M_P acts on M_P/K_P on the left, but this action is not cocompact. We will show that $\mu_{\mathcal{E}}(x)$ and $\mu_{\mathcal{E}}(y)$ lie near an orbit of this action and use this to show that $\rho(y)$ is bounded. Let $B := B_{c_{\phi}+1}(X_P(t_0), A^+)$ be a neighborhood of $X_P(t_0)$ in A^+ , so that $\phi(y) \in B$. Let

$$\beta_P = [PN^+B]_{\mathcal{E}}.$$

If $z \in \mathcal{E}$, $\rho(z) \in P$, and $\phi(z) \in B$, then $z \in \beta_P$; in particular, $x, y \in \beta_P$.

We claim that the image of $\beta_P \cap \mathcal{S}$ is a bounded set in M_P/K_P . If $b \in \beta_P \cap \mathcal{S}$, there is a unique decomposition $b = [n_b a_b]_{\mathcal{E}}$, where $n_b \in N^+$ and $a_b \in B$, and $\mu_{\mathcal{E}}(b) = [\mu(n_b)\mu(a_b)]_{\mathcal{E}}$. Since N^+ is compact, $\mu(n_b)$ is bounded. Since $a_b \in B$, the ratio of two coefficients in a diagonal block of a_b is bounded, and so $\mu(a_b)$ is bounded as well. Thus $\mu_{\mathcal{E}}(\beta_P \cap \mathcal{S})$ is bounded; call this set ω_P .

Since $x \in \beta_P \cap \mathcal{S}$ and $y \in \beta_P \cap \rho(y)\mathcal{S} = \rho(y)(\beta_P \cap \mathcal{S})$, we know $\mu_{\mathcal{E}}(x) \in \omega_P$ and $\mu_{\mathcal{E}}(y) \in m_y \omega_P$. Since $M_P(\mathbb{Z})$ acts properly discontinuously on M_P/K_P and $d_{M_P/K_P}(\mu_{\mathcal{E}}(x), \mu_{\mathcal{E}}(y)) \leq \mathrm{Lip}(\mu_{\mathcal{E}})$, there are only finitely many possibilities for m_y .

To bound n_y , write x and y as $x = n\phi(x)\mathrm{SO}(p)$ and $y = \rho(y)n'\phi(y)\mathrm{SO}(p)$ for some $n, n' \in N^+$. Since $d_{\mathcal{E}}(x, y) \leq 1$, there is a c such that

$$\|(n\phi(x))^{-1}\rho(y)n'\phi(y)\|_2 < c.$$

and thus

$$\begin{aligned} \|\rho(y)\|_2 &< \|n\phi(x)\|_2 \|(n\phi(x))^{-1}\rho(y)n'\phi(y)\|_2 \|(n'\phi(y))^{-1}\|_2 \\ \log \|\rho(y)\|_2 &< \log c + \log \|n\phi(x)\|_2 + \log \|(n'\phi(y))^{-1}\|_2 \\ &= O(d_{\mathcal{E}}(I, \phi(x)) + d_{\mathcal{E}}(I, \phi(y))) \end{aligned}$$

By Corollary 4, we see that $\log \|\rho(y)\|_2 = O(d_{\mathcal{E}}(I, x))$ as desired. \square

The work of Ji and MacPherson [11] suggests how this construction might be extended to lattices in other symmetric spaces. We can replace ϕ with a map from the quotient to the asymptotic cone of the quotient and replace X_P with a generalized Siegel set for P and get similar results.

In the next section, we will need the following corollary, which tells us that if Δ is a 2-cell of τ , then all the edges of Δ satisfy the conditions of Lemma 6 for a single parabolic subgroup P .

Corollary 7. *Let $x_1, x_2, x_3 \in \mathcal{E}$ be such that the distance between any pair of points is at most 1. There is a c'_ρ such that if $\phi(x_1) \in X_P(t_0)$, then for all i, j , we can decompose $\rho(x_i)^{-1}\rho(x_j)$ as a product $\rho(x_i)^{-1}\rho(x_j) = nm$, where $n \in N_P(\mathbb{Z})$, $m \in M_P(\mathbb{Z})$, $d_\Gamma(I, m) < c'_\rho$, and $\|n\|_2 \leq c'_\rho e^{c'_\rho d_\mathcal{E}(x, I)}$.*

In particular, if Δ is a 2-cell in τ , we can choose x to be a vertex of Δ and let $P_\Delta \in \mathcal{P}$ be such that $\phi(f(x)) \in X_{P_\Delta}(t_0)$. Then if y and z are vertices of Δ , then $\bar{f}_0(y)^{-1}\bar{f}_0(z)$ can be decomposed as above.

Proof. This follows from the lemma for $\rho(x_1)^{-1}\rho(x_2)$ and $\rho(x_1)^{-1}\rho(x_3)$, and

$$\rho(x_3)^{-1}\rho(x_2) = (\rho(x_3)^{-1}\rho(x_1))(\rho(x_1)^{-1}\rho(x_2)).$$

□

6. CONSTRUCTING WORDS REPRESENTING EDGES

We will use Lemma 6 to extend \bar{f}_0 to a map $\bar{f}_1 : \tau^{(1)} \rightarrow K_\Gamma$. This corresponds to choosing, for each edge $e = (x, y)$, a word w_e representing $\bar{f}_0(x)^{-1}\bar{f}_0(y)$.

If $\phi(x) \in X_P(t_0)$, we will choose a w_e which is a product of boundedly many generators of M_P and boundedly many words in Σ^* which each represent an elementary matrix in N_P . One difficulty is doing this consistently, so that the boundary of each triangle satisfies this condition for a single P . We will need two main lemmas. The first states that elementary matrices with large coefficients can be represented by “shortcuts”. This is a key ingredient in the proof of the theorem of Lubotzky, Mozes, and Raghunathan [15] which states that when $p \geq 3$, the word metric on Γ is equivalent to the metric induced by the Riemannian metric on G ; see also [18] for an explicit combinatorial construction.

Lemma 8 (see [15]). *If $p \geq 3$, then for every $i, j \in \{1, \dots, p\}$, $i \neq j$, and $x \in \mathbb{Z}$, there is a word $\hat{e}_{ij}(x)$ representing $e_{ij}(x)$ which has length $O(\log |x|)$.*

To state the second lemma, we will need to define some sets of matrix indices. If $P = U(S_1, \dots, S_n)$, let

$$\begin{aligned} \chi(M_P) &:= \{(s_1, s_2) \mid s_1, s_2 \in S_i \text{ for some } i\}, \\ \chi(N_P) &:= \{(s_1, s_2) \mid s_1 \in S_i, s_2 \in S_j \text{ for some } i < j\}, \\ \chi(P) &:= \chi(M_P) \cup \chi(N_P) = \{(s_1, s_2) \mid s_1 \in S_i, s_2 \in S_j \text{ for some } i \leq j\}. \end{aligned}$$

Let P_Δ be as in Corollary 7.

Lemma 9. *If $p \geq 3$, there is a c depending only on p and a choice of a word $w_e \in \Sigma^*$ for each edge e in τ such that if Δ is a 2-cell of τ and $e = (x, y)$ is an edge of Δ , then:*

- w_e represents $\bar{f}_0(x)^{-1}\bar{f}_0(y)$,
- $\ell(w_e) = O(\ell)$,
- w_e can be written as a product $w_e = z_1 \dots z_n$ such that $n \leq c$ and each z_i is either an element of $\Sigma \cap M_{P_\Delta}$ or a word $\hat{e}_{ij}(x)$ where $(i, j) \in \chi(N_{P_\Delta})$ and $|x| \leq c'_\rho e^{c'_\rho \ell}$, where c'_ρ is the constant from Cor. 7.

Proof of Lemma 8. In [15], the $\hat{e}_{ij}(x)$ are constructed by including the solvable group $\mathbb{R} \ltimes \mathbb{R}^2$ in the thick part of G ; since $\mathbb{R}^2 \subset \mathbb{R} \ltimes \mathbb{R}^2$ is exponentially distorted, there are curves in $\mathbb{R} \ltimes \mathbb{R}^2$ which can be approximated by words in Γ . For our purposes, we will need a construction which uses more general solvable groups. In

particular, when $p \geq 4$, we can construct the $\widehat{e}_{ij}(x)$ as approximations of curves in solvable groups with quadratic Dehn function.

Let $S, T \subset \{1, \dots, n\}$ be disjoint subsets and let $s = \#S$ and $t = \#T$. Assume that $s \geq 2$. We will define a solvable subgroup $H_{S,T} \subset U(S, T)$. Let A_1, \dots, A_s be a set of simultaneously diagonalizable positive-definite matrices in $\mathrm{SL}(S, \mathbb{Z})$. The A_i 's have the same eigenvectors; call these shared eigenvectors $v_1, \dots, v_s \in \mathbb{R}^S$, and normalize them to have unit length. The A_i are entirely determined by their eigenvalues, and we can define vectors

$$q_i = (\log \|A_i v_1\|_2, \dots, \log \|A_i v_s\|_2) \in \mathbb{R}^s$$

Since $A_i \in \mathrm{SL}(S, \mathbb{Z})$, the product of its eigenvectors is 1, and the sum of the coordinates of q_i is 0. We require that the A_i are independent in the sense that the q_i span a $(s-1)$ -dimensional subspace of \mathbb{R}^s ; since they are all contained in an $(s-1)$ -dimensional subspace, this is the maximum rank possible. If a set of matrices satisfies these conditions, we call them a set of *independent commuting matrices* for S . A construction of such matrices can be found in Section 10.4 of [7]. The A_i generate a subgroup isomorphic to \mathbb{Z}^{s-1} , and by possibly choosing a different generating set for this subgroup, we can assume that $\lambda_i := \|A_i v_i\|_2 > 1$ for all i .

Let $B_1^{tr}, \dots, B_t^{tr} \in \mathrm{SL}(T, \mathbb{Z})$ (where tr represents the transpose of a matrix) be a set of independent commuting matrices for T and let $w_1, \dots, w_t \in \mathbb{R}^T$ be the basis of unit eigenvectors of the B_i^{tr} . Choose the B_i so that $\mu_i := \|w_i B_i\|_2 > 1$. Let

$$\begin{aligned} H_{S,T} &:= \left\{ \begin{pmatrix} \prod_i A_i^{x_i} & V \\ 0 & \prod_i B_i^{y_i} \end{pmatrix} \mid x_i, y_i \in \mathbb{R}, V \in \mathbb{R}^S \otimes \mathbb{R}^T \right\} \\ &= (\mathbb{R}^{s-1} \times \mathbb{R}^{t-1}) \ltimes (\mathbb{R}^S \otimes \mathbb{R}^T). \end{aligned}$$

Note that $H_{S,T} \cap \Gamma$ is a cocompact lattice in $H_{S,T}$, so $H_{S,T}$ is contained in the thick part of G . That is, if ϵ is sufficiently small, then $[H_{S,T}]_{\mathcal{E}} \subset \mathcal{E}(\epsilon)$, so Lemma 1 can be used to construct words in Σ^* out of paths in $H_{S,T}$. We will use this and the fact that the subgroup $\mathbb{R}^S \otimes \mathbb{R}^T$ is exponentially distorted in $H_{S,T}$ to get short words in Σ^* representing certain unipotent matrices.

By abuse of notation, let A_i and B_i refer to the corresponding matrices in $H_{S,T}$. The group $H_{S,T}$ is generated by powers of the A_i , powers of the B_i , and elementary matrices in the sense that any element of $H_{S,T}$ can be written as

$$\prod A_i^{x_i} \prod B_i^{y_i} \begin{pmatrix} I_S & V \\ 0 & I_T \end{pmatrix},$$

for some $x_i, y_i \in \mathbb{R}$ and $V \in \mathbb{R}^S \otimes \mathbb{R}^T$, where I_S and I_T represent the identity matrix in $\mathrm{SL}(S, \mathbb{Z})$ and $\mathrm{SL}(T, \mathbb{Z})$ respectively. As with discrete groups we will associate generators with curves, and words with concatenations of curves. We let A_i^x correspond to the curve

$$d \mapsto \begin{pmatrix} A_i^{xd} & 0 \\ 0 & I_T \end{pmatrix},$$

B_i^x to the curve

$$d \mapsto \begin{pmatrix} I_S & 0 \\ 0 & B_i^{xd} \end{pmatrix},$$

and

$$u(V) = \begin{pmatrix} I_S & V \\ 0 & I_T \end{pmatrix}$$

to the curve

$$d \mapsto \begin{pmatrix} I_S & dV \\ 0 & I_T \end{pmatrix},$$

where in all cases, d ranges from 0 to 1. Let $c \geq \max\{\ell(A_i), \ell(B_i)\}$. Then the word $A_i^x u(v_i \otimes w) A_i^{-x}$ represents the matrix $u(\lambda_i^x v_i \otimes w)$ and corresponds to a curve of length at most $2cx + \|v_i\|_2 \|w\|_2$ connecting I and $u(\lambda_i^x v_i \otimes w)$. Similarly, if $t \geq 2$, then $B_i^{-x} u(v \otimes w_i) B_i^x$ has length at most $2cx + \|v_i\|_2 \|w\|_2$ and connects I and $u(\mu_i^x v \otimes w_i)$.

If $V \in \mathbb{R}^S \otimes \mathbb{R}^T$, then

$$V = \sum_{i,j} x_{ij} v_i \otimes w_j$$

for some $x_{ij} \in \mathbb{R}$. Let

$$l_i(x) = \begin{cases} \lceil \log_{\lambda_i} |x| \rceil & \text{if } |x| > 1, \\ 0 & \text{if } |x| \leq 1, \end{cases}$$

and define

$$\gamma_{ij}(x) = A_i^{l_i(x)} u\left(\frac{x}{\lambda_i^{l_i(x)}} v_i \otimes w_j\right) A_i^{-l_i(x)}.$$

Note that $|x/\lambda_i^{l_i(x)}| \leq 1$. Let

$$\hat{u}(V) := \prod_{i,j} \gamma_{ij}(x_{ij}).$$

Then $\hat{u}(V)$ represents $u(V)$ and there is a c' such that

$$\ell(\hat{u}(V)) \leq c'(1 + \log \|V\|_2)$$

for all V .

If $i \in S$ and $j \in T$, then $e_{ij}(x) = u(xz_i \otimes z_j) \in H_{S,T}$. If $x \in \mathbb{Z}$, then we can apply Lemma 1 to approximate $\hat{u}(xz_i \otimes z_j)$ by a word $\hat{e}_{ij;S,T}(x) \in \Sigma^*$ which represents $e_{ij}(x)$ and whose length is $O(\log |x|)$. In general, changing S and T will change $\hat{e}_{ij;S,T}(x)$ drastically, but later, we will prove that if $i \in S, S'$ and $j \in T, T'$, and S and S' satisfy some mild conditions, then $\hat{e}_{ij;S,T}(x)$ and $\hat{e}_{ij;S',T'}(x)$ are connected by a homotopy of area $O((\log |x|)^2)$. Because of this, the choice of S and T is largely irrelevant. Thus, for each (i, j) , we choose a $d \notin \{i, j\}$ and let

$$\hat{e}_{ij}(x) = \hat{e}_{ij;\{i,d\},\{j\}}(x).$$

□

Proof of Lemma 9. If $e = (x, y)$ is an interior edge of τ , it is in the boundary of two 2-cells; call these Δ and Δ' . By Corollary 7, there is a c depending only on p such that if $g = \bar{f}_0(x)^{-1} \bar{f}_0(y) \in \Gamma$ and g_{ij} is the (i, j) -coefficient of g , then

$$\begin{aligned} g &\in P_\Delta(\mathbb{Z}) \cup P_{\Delta'}(\mathbb{Z}) \\ |g_{ij}| &< c \quad \text{if } (i, j) \in \chi(M_{P_\Delta}) \cup \chi(M_{P_{\Delta'}}) \\ \|g\|_\infty &< ce^{c\ell} \end{aligned}$$

The last inequality follows from the fact that $d_{\mathcal{E}}([I]_{\mathcal{E}}, f(x)) \leq \ell$. Note that $P_\Delta \cap P_{\Delta'}$ is parabolic.

We express g as a word in Σ^* as follows. Let $g = nm$, where $n \in N_{P_\Delta \cap P_{\Delta'}}(\mathbb{Z})$ and $m \in M_{P_\Delta \cap P_{\Delta'}}(\mathbb{Z})$. Then $\|m\|_\infty < c$, and there is a c' depending on p such that $\|m\|_2 < c'$ and $\|m^{-1}\|_2 < c'$. Therefore,

$$\|n\|_\infty \leq \|gm^{-1}\|_2 \leq p^2 c' c e^{c\ell}$$

and if $(i, j) \in \chi(M_{P_\Delta}) \cup \chi(M_{P_{\Delta'}})$, then $|n_{ij}| < pc'$.

Since n is a unipotent matrix, we can write n as a product

$$n = \prod_{(i,j) \in \chi(N_{P_\Delta \cap P_{\Delta'}})} e_{ij}(n_{ij})$$

for an appropriate ordering of $\chi(N_{P_\Delta \cap P_{\Delta'}})$. We can replace the terms corresponding to large coefficients with shortcuts. Let

$$w_1 = \prod_{(i,j) \in \chi(N_{P_\Delta \cap P_{\Delta'}})} \begin{cases} e_{ij}^{n_{ij}} & \text{if } (i, j) \in \chi(M_{P_\Delta}) \cup \chi(M_{P_{\Delta'}}), \\ \widehat{e}_{ij}(n_{ij}) & \text{otherwise.} \end{cases}$$

This represents n and has length $O(\ell)$.

Finally, there is a c'' depending only on p such that we can write m as a product $w_2 \in (\Sigma \cap M_{P_\Delta \cap P_{\Delta'}})^*$ of no more than c'' generators of $M_{P_\Delta \cap P_{\Delta'}}$. Let

$$\bar{f}_1(e) = w_1 w_2 \in \Sigma^*.$$

This satisfies the conditions of the lemma for both Δ and Δ' .

If e is on the boundary of τ and e is an edge of Δ , then $P_\Delta = G$, and since $N_G = \{I\}$, there is a c such that $d_\Gamma(\bar{f}_0(x), \bar{f}_0(y)) < c$. We can take w_e to be a geodesic word representing $\bar{f}_0(x)^{-1} \bar{f}_0(y)$. \square

We then construct \bar{f}_1 by defining $\bar{f}_1|_e$ to be the curve corresponding to w_e . Note that $\bar{f}_1|_{\partial\tau}$ differs from the original w by only a bounded distance. In particular, there is an annulus in K_Γ whose boundary curves are w and $\bar{f}_1|_{\partial\tau}$ and which has area $O(\ell)$.

7. FILLING THE 2-SKELETON

In the previous section, we reduced the problem of filling α to the problem of filling the curves $\bar{f}_1(\partial\Delta)$, where Δ ranges over all 2-cells of τ . Each of these curves is a product of a bounded number of elements of Σ and a bounded number of shortcuts $\widehat{e}_{ij}(x)$. In this section, we will describe methods for filling such curves. The key to many of these methods is the group $H_{S,T}$ from Section 6, which we used to construct \widehat{e}_{ij} . This group has two key properties. First, when either S or T is large enough, then $H_{S,T}$ has quadratic Dehn function; this is a special case of a theorem of de Cornulier and Tessera. Second, when both S and T are sufficiently large, $H_{S,T}$ contains multiple ways to shorten elementary matrices. A good choice of shortening makes it possible to fill many discs, including discs corresponding to the Steinberg relations.

We first state a special case of a theorem of de Cornulier and Tessera:

Theorem 4 ([4]). *If $s \geq 3$ or $t \geq 3$, then $H_{S,T}$ has quadratic Dehn function.*

The quadratic Dehn function will let us switch between different shortenings. Say $\#S \geq 3$, $\#T \geq 2$, and let $A_i \in \mathrm{SL}(S, \mathbb{Z})$, $B_i \in \mathrm{SL}(T, \mathbb{Z})$, $v_i \in \mathbb{R}^S$, and $w_i \in \mathbb{R}^T$ be as in Section 6. Then we can express $u(xv_i \otimes w_j)$ either as $A_i^k u(v_i \otimes w_j) A_i^{-k}$ or as $B_j^{-l} u(v_i \otimes w_j) B_j^l$. In the following lemma, we switch between these representations

to find fillings for words representing conjugates of $\widehat{u}(V)$. Let $\Sigma_S := \Sigma \cap \mathrm{SL}(S, \mathbb{Z})$ and $\Sigma_T := \Sigma \cap \mathrm{SL}(T, \mathbb{Z})$. These are generating sets for $\mathrm{SL}(S, \mathbb{Z})$ and $\mathrm{SL}(T, \mathbb{Z})$.

Lemma 10. *If $\#S \geq 3$ and $\#T \geq 2$ or vice versa, there is an $\epsilon > 0$ and a $c > 0$ such that if γ is a word in $(\Sigma_S \cup \Sigma_T)^*$ representing $(M, N) \in \mathrm{SL}(S, \mathbb{Z}) \times \mathrm{SL}(T, \mathbb{Z})$, then*

$$\delta_{\mathcal{E}(\epsilon)}([\gamma \widehat{u}(V) \gamma^{-1}]_{\mathcal{E}}, [\widehat{u}(MVN^{-1})]_{\mathcal{E}}) = c(\ell(\gamma) + \log(\|V\|_2 + 2))^2.$$

Proof. Let $\omega := \gamma \widehat{u}(V) \gamma^{-1}]_{\mathcal{E}} \widehat{u}(MVN^{-1})^{-1}$; this is a closed curve in G .

We first consider the case that $V = xv_i \otimes w_j$ and $\gamma \in \Sigma_T^*$. In this case, $M = I$; and $\gamma \widehat{u}(V) \gamma^{-1}$ and $\widehat{u}(VN^{-1})$ are both words in the group

$$\begin{aligned} F &:= \left\{ \left(\begin{pmatrix} \prod_i A_i^{x_i} & V \\ 0 & D \end{pmatrix} \middle| x_i \in \mathbb{R}, D \in \mathrm{SL}(T, \mathbb{Z}), V \in \mathbb{R}^S \otimes \mathbb{R}^T \right\} \\ &= (\mathbb{R}^{s-1} \times \mathrm{SL}(T, \mathbb{Z})) \ltimes (\mathbb{R}^S \otimes \mathbb{R}^T). \end{aligned}$$

This group is generated by

$$\Sigma_F := \{A_i^x \mid x \in \mathbb{R}\} \cup \{u(V) \mid V \in \mathbb{R}^S \otimes \mathbb{R}^T\} \cup \Sigma_T.$$

Let $\epsilon \leq 1/2$ be sufficiently small that $H_{S,T} \subset G(\epsilon)$. Since $G(\epsilon)$ is contractible and $F \subset G(\epsilon)$, words in Σ_F^* correspond to curves in $G(\epsilon)$. We will show that

$$\delta_{G(\epsilon)}(\gamma \widehat{u}(V) \gamma^{-1}, \widehat{u}(VN^{-1})) \leq O(\ell(\omega)^2).$$

Words in Σ_F^* satisfy certain relations which correspond to discs in $G(\epsilon)$. In particular, note that if $\sigma \in \Sigma_T$, $|x| \leq 1$, and $\|W\|_2 \leq 1$, then

$$(2) \quad [\sigma, A_k^x]$$

and

$$(3) \quad \sigma u(W) \sigma^{-1} u(W \sigma^{-1})^{-1}$$

are both closed curves of bounded length. Since $G(\epsilon)$ is contractible, their filling areas are bounded, and we can think of them as “relations” in F .

Let $C = \log_{\min_k \{\lambda_k\}}(p+1)$, and let $z = C\ell(\gamma) + l_i(x)$. This choice of z ensures that

$$\|\lambda_i^{-z} V N\|_2 \leq 1.$$

Indeed, it ensures that if $d_{\mathrm{SL}(T, \mathbb{Z})}(I, N') \leq \ell(\gamma)$, then

$$\|\lambda_i^{-z} V N'\|_2 \leq 1.$$

Furthermore, $z = O(\ell(\omega))$.

We will construct a homotopy which lies in $G(\epsilon)$ and goes through the stages

$$\begin{aligned} \omega_1 &= \gamma \widehat{u}(V) \gamma^{-1} \\ \omega_2 &= \gamma A_i^z u(\lambda_i^{-z} V) A_i^{-z} \gamma^{-1} \\ \omega_3 &= A_i^z \gamma u(\lambda_i^{-z} V) \gamma^{-1} A_i^{-z} \\ \omega_4 &= A_i^z u(\lambda_i^{-z} V N^{-1}) A_i^{-z} \\ \omega_5 &= \widehat{u}(V N^{-1}). \end{aligned}$$

Each stage is a word in Σ_F^* and so corresponds to a curve in $G(\epsilon)$.

We can construct a homotopy between ω_1 and ω_2 and between ω_4 and ω_5 using Thm. 4. We need to construct homotopies between ω_2 and ω_3 and between ω_3 and ω_4 .

We can transform ω_2 to ω_3 by applying (2) at most $O(\ell(\omega)^2)$ times. This corresponds to a homotopy with area $O(\ell(\omega)^2)$. Similarly, we can transform ω_3 to ω_4 by applying (3) at most $O(\ell(\omega))$ times, corresponding to a homotopy of area $O(\ell(\omega))$. Combining all of these homotopies, we find that

$$\delta_{G(\epsilon)}(\gamma \hat{u}(V) \gamma^{-1}, \hat{u}(VN^{-1})) \leq O(\ell(\omega)^2).$$

as desired.

We can use this case to generalize to the case $V = \sum_{i,j} x_{ij} v_i \otimes w_j$ and $\gamma \in \Sigma_T^*$. By applying the case to each term of $\hat{u}(V)$, we obtain a homotopy of area $O(\ell(\omega)^2)$ from $\gamma \hat{u}(V) \gamma^{-1}$ to

$$\prod_{i,j} \hat{u}(x_{ij} v_i \otimes w_j N^{-1}).$$

This is a curve in $H_{S,T}$ of length $O(\ell(\omega))$ which connects I and $u(VN^{-1})$. By Thm. 4, there is a homotopy between this curve and $\hat{u}(VN^{-1})$ of area $O(\ell(\omega)^2)$.

When $\gamma \in \Sigma_S^*$, we instead let F be the group

$$\begin{aligned} F &:= \left\{ \begin{pmatrix} D & V \\ 0 & \prod_i B_i^{x_i} \end{pmatrix} \middle| x_i \in \mathbb{R}, D \in \mathrm{SL}(S, \mathbb{Z}), V \in \mathbb{R}^S \otimes \mathbb{R}^T \right\} \\ &= (\mathrm{SL}(S, \mathbb{Z}) \times \mathbb{R}^{t-1}) \ltimes (\mathbb{R}^S \otimes \mathbb{R}^T). \end{aligned}$$

Here, $\hat{u}(V)$ is not a word in F , but since $\#T \geq 2$, we can replace the A_i with the B_i in the construction of $\hat{u}(V)$. This results in shortcuts $\hat{u}'(V)$ in the alphabet

$$\{B_i^x \mid x \in \mathbb{R}\} \cup \{u(V) \mid V \in \mathbb{R}^S \otimes \mathbb{R}^T\}.$$

These are curves in $H_{S,T}$ which represent $u(V)$ and have length $O(\log \|V\|_2)$, so by Thm. 4, there is a homotopy of area $O((\log \|V\|_2)^2)$ between $\hat{u}'(V)$ and $\hat{u}(V)$.

The argument for $\gamma \in \Sigma_S^*$ shows that

$$\delta_{G(\epsilon)}(\gamma \hat{u}'(V) \gamma^{-1}, \hat{u}'(MV)) = O(\ell(\omega)^2).$$

Replacing $\hat{u}'(V)$ with $\hat{u}(V)$ and $\hat{u}'(MV)$ with $\hat{u}(MV)$ adds area $O(\ell(\omega)^2)$, so

$$\delta_{G(\epsilon)}(\gamma \hat{u}(V) \gamma^{-1}, \hat{u}(MV)) = O(\ell(\omega)^2).$$

If $\gamma \in (\Sigma_S \cup \Sigma_T)^*$, and $\gamma_S \in \Sigma_S^*$ and $\gamma_T \in \Sigma_T^*$ are the words obtained by deleting all the letters in Σ_T and Σ_S respectively, then $\delta_G(\gamma, \gamma_S \gamma_T) = O(\ell(\omega)^2)$. We can construct a homotopy from $\gamma \hat{u}(V) \gamma^{-1}$ to $\hat{u}(MVN^{-1})$ going through the steps

$$\begin{aligned} \gamma \hat{u}(V) \gamma^{-1} &\rightarrow \gamma_S \gamma_T \hat{u}(V) \gamma_T^{-1} \gamma_S^{-1} \\ &\rightarrow \gamma_S \hat{u}(VN^{-1}) \gamma_S^{-1} \\ &\rightarrow \hat{u}(MVN^{-1}). \end{aligned}$$

This homotopy has area $O(\ell(\omega)^2)$. \square

Recall that $\hat{e}_{ij;S,T}(x)$ is an approximation of a curve $\hat{u}(xz_i \otimes z_j)$; we write this curve as $\hat{u}_{S,T}(xz_i \otimes z_j)$ to distinguish curves in different solvable subgroups.

Lemma 11. *If $p \geq 5$, $i \in S, S'$ and $j \in T, T'$, where $2 \leq \#S, \#S' \leq p-2$, then*

$$\delta_\Gamma(\hat{e}_{ij;S,T}(x), \hat{e}_{ij;S',T'}(x)) = O((\log |x|)^2).$$

Proof. Case 1: Let $V = xz_i \otimes z_j$. We first consider the case that $S = S'$. Both $\widehat{u}_{S,T}(V)$ and $\widehat{u}_{S',T'}(V)$ are curves in H_{S,S^c} for S^c the complement of S . Since $k \geq 5$, Thm. 4 states that H_{S,S^c} has quadratic Dehn function, so the lemma follows. In particular,

$$\delta_\Gamma(\widehat{e}_{ij;S,T}(x), \widehat{e}_{ij;S,\{j\}}(x)) = O((\log |x|)^2).$$

Case 2: Let $S \subset S'$, $\#S' \geq 3$, $T \subset T'$, and $\#T' \geq 2$. Let $\{A_i\}$ be as in the definition of $H_{S,T}$, with eigenvectors v_i and let $\{A'_i\} \in \text{SL}(S', \mathbb{Z})$ be the set of independent commuting matrices used in defining $H_{S',T'}$. Recall that $\widehat{u}_{S,T}(V)$ is the concatenation of curves γ_i of the form

$$A_i^{c_i} u(x_i v_i \otimes z_j) A_i^{-c_i}$$

where $c_i \in \mathbb{Z}$ and $|x_i| \leq 1$. Since $A_i \in \text{SL}(S, \mathbb{Z}) \subset \text{SL}(S', \mathbb{Z})$, each of these curves satisfies the hypotheses of Lemma 10 for S' and T' , and so there is a homotopy of area $O((\log |x|)^2)$ between γ_i and

$$\widehat{u}_{S',T'}(\lambda_i^{c_i} x_i v_i \otimes z_j).$$

Each of these curves lie in $H_{S',T'}$, and since $\widehat{u}_{S',T'}(V)$ also lies in $H_{S',T'}$ and $H_{S',T'}$ has quadratic Dehn function,

$$\delta_\Gamma(\widehat{e}_{S,T}(V), \widehat{e}_{S',T'}(V)) = O((\log |x|)^2).$$

Combining these two cases proves the lemma. First, we construct a homotopy between $\widehat{e}_{S,T}(V)$ and a word of the form $\widehat{e}_{\{i,d\},\{j\}}(V)$. If $\#S = 2$, we can use case 1. Otherwise, let $d \in S$ be such that $d \neq i$. We can construct a homotopy going through the stages

$$\widehat{e}_{S,T}(V) \rightarrow \widehat{e}_{S,S^c}(V) \rightarrow \widehat{e}_{\{i,d\},\{j\}}(V).$$

The second step is an application of case 2, possible because $\{i,d\} \subset S$, $\#S \geq 3$, and $\{j\} \subset S^c$.

Similarly, we can construct a homotopy between $\widehat{e}_{S,T}(V)$ and a word of the form $\widehat{e}_{\{i,d'\},\{j\}}(V)$. If $d = d'$, we're done. Otherwise, we can use case 2 to construct homotopies between each word and $\widehat{e}_{\{i,d,d'\},\{i,d,d'\}^c}(V)$. \square

Using these lemmas, we can give fillings for a wide variety of curves; note that (1)–(3) are versions of the Steinberg relations.

Lemma 12. *If $p \geq 5$ and $x, y \in \mathbb{Z} - \{0\}$, then*

(1) *If $1 \leq i, j \leq p$ and $i \neq j$, then*

$$\delta_\Gamma(\widehat{e}_{ij}(x)\widehat{e}_{ij}(y), \widehat{e}_{ij}(x+y)) = O((\log |x| + \log |y|)^2).$$

In particular,

$$\delta_\Gamma(\widehat{e}_{ij}(x)\widehat{e}_{ij}(-x)) = O((\log |x|)^2).$$

(2) *If $1 \leq i, j, k \leq p$ and $i \neq j \neq k$, then*

$$\delta_\Gamma([\widehat{e}_{ij}(x), \widehat{e}_{jk}(y)], \widehat{e}_{ik}(xy)) = O((\log |x| + \log |y|)^2).$$

(3) *If $1 \leq i, j, k, l \leq p$, $i \neq l$, and $j \neq k$*

$$\delta_\Gamma([\widehat{e}_{ij}(x), \widehat{e}_{kl}(y)]) = O((\log |x| + \log |y|)^2).$$

(4) Let $1 \leq i, j, k, l \leq p$, $i \neq j$, and $k \neq l$, and

$$s_{ij} = e_{ji}^{-1} e_{ij} e_{ji}^{-1},$$

so that s_{ij} represents

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{SL}(\{i, j\}, \mathbb{Z}).$$

Then

$$\delta_{\Gamma}(s_{ij} \hat{e}_{kl}(x) s_{ij}^{-1}, \hat{e}_{\sigma(k)\sigma(l)}(\tau(k, l)x)) = O((\log |x| + \log |y|)^2),$$

where σ is the permutation switching i and j , and $\tau(k, l) = -1$ if $k = i$ or $l = i$ and 1 otherwise.

(5) If $b = \mathrm{diag}(b_1, \dots, b_p)$, then

$$\delta_{\Gamma}(b \hat{e}_{ij}(x) b^{-1}, \hat{e}_{ij}(b_i b_j x)(\tau(k, l)x)) = O(\log |x|^2).$$

Proof. For part 1, note that

$$\hat{e}_{ij}(x) \hat{e}_{ij}(y) \hat{e}_{ij}(x + y)^{-1}$$

is within bounded distance of a closed curve in $H_{\{j\}^c, \{j\}}$ of length $O(\log |x|)$. Thus part 1 of the lemma follows from Thm. 4.

For part 2, let $d \notin \{i, j, k\}$ and let $S = \{i, j, d\}$, so that $\hat{e}_{ij; \{i, d\}, \{j\}}(x)$ is a word in $\mathrm{SL}(S, \mathbb{Z})$. We construct a homotopy going through the stages

$$\begin{aligned} & [\hat{e}_{ij}(x), \hat{e}_{jk}(y)] \hat{e}_{ik}(xy)^{-1} \\ & [\hat{e}_{ij; \{i, d\}, \{j\}}(x), \hat{u}_{S, \{k\}}(yz_j \otimes z_k)] \hat{e}_{ik; S, \{k\}}(xy)^{-1} && \text{by Lem. 11} \\ & \hat{u}_{S, \{k\}}((xyz_i + yz_j) \otimes z_k) \hat{u}_{S, \{k\}}(yz_j \otimes z_k)^{-1} \hat{e}_{ik; S, \{k\}}(xyz_i \otimes z_k)^{-1} && \text{by Lem. 10} \\ & \varepsilon && \text{by Thm. 4} \end{aligned}$$

All these homotopies have area $O((\log |x| + \log |y|)^2)$.

For part 3, we let $S = \{i, j, d\}$, $T = \{k, l\}$, and use the same techniques to construct a homotopy going through the stages

$$\begin{aligned} & [\hat{e}_{ij}(x), \hat{e}_{kl}(y)] \\ & [\hat{e}_{ij; S, T}(x), \hat{e}_{kl; S, T}(y)] && \text{by Lem. 11} \\ & \varepsilon && \text{by Thm. 4} \end{aligned}$$

This homotopy has area $O((\log |x| + \log |y|)^2)$.

Part 4 breaks into several cases depending on k and l . When i, j, k , and l are distinct, the result follows from part 3, since $s_{ij} = e_{ji}^{-1} e_{ij} e_{ji}^{-1}$, and we can use part 3 to commute each letter past $\hat{e}_{kl}(x)$. If $k = i$ and $l \neq j$, let $d, d' \notin \{i, j, l\}$, $d \neq d'$, and let $S = \{i, j, d\}$ and $T = \{l, d'\}$. There is a homotopy from

$$s_{ij} \hat{e}_{il}(x) s_{ij}^{-1} \hat{e}_{jl}(-x)^{-1}$$

to

$$s_{ij} \hat{u}_{S, T}(xz_i \otimes z_l) s_{ij}^{-1} \hat{e}_{jl}(-xz_j \otimes z_l)$$

of area $O((\log |x|)^2)$, and since $s_{ij} \in \Sigma_S^*$, the proposition follows by an application of Lemma 10. A similar argument applies to the cases $k = j$ and $l \neq i$; $k \neq i$ and $l = j$; and $k \neq j$ and $l = i$.

If $(k, l) = (i, j)$, let $d, d' \notin \{i, j\}$. There is a homotopy going through the stages

$$\begin{array}{ll}
s_{ij}\widehat{e}_{ij}(x)s_{ij}^{-1} & \\
s_{ij}[e_{id}, \widehat{e}_{dj}(x)]s_{ij}^{-1} & \text{by part 2} \\
[s_{ij}e_{id}s_{ij}^{-1}, s_{ij}\widehat{e}_{dj}(x)s_{ij}^{-1}] & \text{by free insertion} \\
[e_{jd}^{-1}, \widehat{e}_{di}(x)] & \text{by previous cases} \\
\widehat{e}_{jd}(-x) & \text{by part 2}
\end{array}$$

and this homotopy has area $O((\log |x|)^2)$. One can treat the case $(k, l) = (j, i)$ the same way.

Since any diagonal matrix in Γ is the product of at most p elements s_{ij} , part 5 follows from part 4. \square

This lemma allows us to fill shortenings of curves in nilpotent subgroups of Γ efficiently.

Lemma 13. *Let $P = U(S_1, \dots, S_s) \in \mathcal{P}$, let $w_i = \widehat{e}_{a_i b_i}(x_i)$ and let $w = w_1 \dots w_d$ for some $(a_i, b_i) \in \chi(N_P)$. Let $h = \max\{\log |x_i|, 1\}$. If w represents the identity, then $\delta_G(w) = O(d^3 h^2)$.*

Proof. We first describe a normal form for elements of N_P . Let

$$\chi_k(N_P) = \{(a, b) \mid a \in S_k, (a, b) \in \chi(N_P)\}.$$

The set $\{e_{ab} \mid (a, b) \in \chi_k(N_P)\}$ generates an abelian subgroup of Γ . If $n \in N_P$, let n_{ab} be the (a, b) -coefficient of n and let

$$\kappa_q(n) = \prod_{(a, b) \in \chi_q(N_P)} \widehat{e}_{ab}(n_{ab}).$$

Let

$$\nu_P(n) = \kappa_s(n)\kappa_{s-1}(n) \dots \kappa_1(n)$$

This is a word representing n , and it has length $O(\log \|n\|_2)$.

Let $n_i \in \Gamma$ be the element represented by $w_1 \dots w_i$. There is a c such that $\log \|n_i\|_2 \leq chd$. The words $\nu_P(n_i)$ connect the identity to points on w , so we can fill w by filling the wedges $\nu_P(n_{i-1})w_i\nu_P(n_i)^{-1}$; we consider this filling as a homotopy between $\nu_P(n_{i-1})w_i$ and $\nu_P(n_i)$. Note that if $a_i \in S_k$, then

$$\kappa_s(n_{i-1}) \dots \kappa_{k+1}(n_{i-1}) = \kappa_s(n_i) \dots \kappa_{k+1}(n_i),$$

so it suffices to transform

$$\kappa_k(n_{i-1}) \dots \kappa_1(n_{i-1})w_i \rightarrow \kappa_k(n_i) \dots \kappa_1(n_i).$$

We can use parts 2 and 3 of Lemma 12 to move w_i to the left. That is, we repeatedly replace subwords of the form $\widehat{e}_{ab}(x)w_i$ with $w_i\widehat{e}_{ab}(x)$ if $b \neq a_i$ and with $w_i\widehat{e}_{ab_i}(xx_i)\widehat{e}_{ab}(x)$ if $b = a_i$. We always have $a \in S_j$ for some $j \leq k$, so $a < b_i$.

Each step has cost $O((\log |x| + \log |x_i|)^2)$. Since $\log |x| \leq \log \|n_i\|_2 \leq chd$, this is $O(h^2 d^2)$. We repeat this process until we have moved w_i to the left end of the word, which takes at most p^2 steps and has total cost $O(h^2 d^2)$. The result is a word of the form $\kappa'_k \dots \kappa'_1$ where κ'_q is a product of words of the form $\widehat{e}_{ab}(x)$ for $(a, b) \in \chi_q(N_P)$. Furthermore, the κ'_q are obtained from the $\kappa_q(n_i)$ by inserting at most p^2 additional words in all (at most one word is added in each step, in addition to the original w_i).

Since the elements represented by the terms of κ'_q all commute, we can use parts 1 and 3 of Lemma 12 to rearrange the terms in each κ'_q and transform $\kappa'_k \dots \kappa'_1$ into $\kappa_k(n_i) \dots \kappa_1(n_i)$. This takes at most $4p^4$ applications of part 3 and at most $2p^2$ applications of part 1, each of which has cost $O(h^2 d^2)$. Thus

$$\delta_\Gamma(\kappa'_k \dots \kappa'_1, \kappa_k(n_i) \dots \kappa_1(n_i)) = O(h^2 d^2),$$

and so

$$\delta_\Gamma(\nu_P(n_{i-1})w_i\nu_P(n_i)^{-1}) = O(h^2 d^2).$$

To fill w , we need to fill d such wedges, so $\delta_\Gamma(w) = O(h^2 d^3)$. \square

In particular, if d is fixed, then $\delta_\Gamma(w) = O(h^2)$.

Finally, we use these tools to fill the curves that occur as $\bar{f}_1(\partial\Delta)$.

Lemma 14. *If Δ is a 2-cell in τ ,*

$$\delta_{K_\Gamma}(\bar{f}_1(\partial\Delta)) = O(\ell^2).$$

Proof. By Lemma 9, there is a c depending only on p such that we can write the word corresponding to $\bar{f}_1(\partial\Delta)$ as $g = g_1 \dots g_d$, where $d \leq c$ and each g_i is either an element of $\Sigma \cap M_{P_\Delta}$ or a word $\hat{e}_{ab}(x)$ where $(a, b) \in \chi(N_{P_\Delta})$ and $|x| \leq ce^{c\ell}$.

Let $x_i \in P$ be the element represented by $g_1 \dots g_i$; by the hypotheses, there is a c' independent of α such that $\|x_i\|_2 \leq c'e^{c'\ell}$. Let $x_i = m_i n_i$ for some $m_i \in M_P$ and $n_i \in N_P$. Then $d_\Gamma(I, m_i) \leq c$, and there is a c'' independent of α such that $\|n_i\|_2 \leq c''e^{c''\ell}$.

Let γ_i be a geodesic word representing m_i , and let $w_i = \gamma_i \nu_P(n_i)$. The w_i are words of length $O(\ell(\alpha))$ connecting points on g to the identity, and we can get a filling of g by filling the wedges $w_i g_{i+1} w_{i+1}^{-1}$.

The filling depends on g_{i+1} . If $g_{i+1} \in \Sigma_{M_P}$, then

$$w_i g_{i+1} w_{i+1}^{-1} = \gamma_i \nu_P(n_i) g_{i+1} \nu_P(n_{i+1})^{-1} \gamma_{i+1}^{-1},$$

and $g_{i+1}^{-1} n_i g_{i+1} = n_{i+1}$. Lemma 12 allows us to move g_{i+1} past the individual terms of $\nu_P(n_i)$, using $O(\ell(\alpha)^2)$ steps. After this, we have a word of the form

$$\gamma_i g_{i+1} h_1 \dots h_k \nu_P(n_{i+1})^{-1} \gamma_{i+1}^{-1},$$

where $h_i = \hat{e}_{a_i b_i}(x_i)$ for some $(a_i, b_i) \in \chi(N_P)$, $|x_i| \leq c''e^{\ell(\alpha)}$, and $k \leq p^2$. By Lemma 13, $h_1 \dots h_k \nu_P(n_{i+1})^{-1}$ can be reduced to the trivial word at cost $O(\ell^2)$. This leaves us with the word $\gamma_i g_{i+1} \gamma_{i+1}^{-1}$; this has length at most $2c + 1$ and can be reduced to the trivial word at bounded cost.

If $g_{i+1} = \hat{e}_{ab}(x)$ for $(a, b) \in \chi(N_P)$, then $\gamma_i = \gamma_{i+1}$, and $\nu_P(n_i) g_{i+1} \nu_P(n_{i+1})^{-1}$ represents the identity. This satisfies the hypotheses of Lemma 13, and can be reduced to the trivial word at cost $O(\ell(\alpha)^2)$. This leaves $\gamma_i \gamma_{i+1}^{-1}$; as before, this has length at most $2c$ and can thus be reduced to the trivial word at bounded cost.

Thus the cost of filling each wedge is $O(\ell^2)$. Since there are at most c wedges, the cost of filling w is $O(\ell^2)$. \square

Since there are $O(\ell^2)$ such 2-cells to fill, we can fill $\bar{f}_1(\partial\tau)$ with area $O(\ell^4)$. Furthermore, $\bar{f}_1(\partial\tau)$ is a bounded distance from w in K_Γ , so

$$\delta_\Gamma(w) \leq \delta_{K_\Gamma}(w, \bar{f}_1(\partial\tau)) + \delta_{K_\Gamma}(\bar{f}_1(\partial\tau)) = O(\ell^4).$$

This proves Theorem 1.

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